Electrostatic potential energy stored in a hemispherical surface with uniform surface charge distribution

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(Dated: March 11, 2021)

We report an exact result for the electrostatic potential energy stored in a hemispherical surface with uniform surface charge density. This system lacks the spherical symmetry of a uniformly charged spherical surface or its solid sphere counterpart. Consequently, a hemispherical surface with all its charge uniformly spread on its surface represents an interesting case study in electrostatics. In this work we combine the fact that the body has axial symmetry with clever mathematical tools to obtain an outcome that is remarkably simple. The final exact expression obtained for the electrostatic potential energy stored in a hemispherical surface with uniform surface charge density answers a long-standing question. Though the bulk of the treatment is analytical, the accuracy of the findings can also be ascertained by using symbolic computation software or numerical recipes. Hence, the present analytical results can be compared to ones obtained with various powerful numerical methods which nowadays are routinely used to check the correctness of analytical and semi-analytical calculations.

Keywords: Electrostatic potential energy, Electrostatic self-energy, Uniform surface charge density, Hemispherical surface.

I. INTRODUCTION

In electrostatics, self-energy of a particular charge distribution is the energy required to assemble the charges from infinity to that particular configuration. This quantity represents the electrostatic potential energy stored in the system of charges[1–6]. The calculation of the total electrostatic potential energy stored in a body with continuous charge distribution is generally very difficult if the charged body has an arbitrary shape or form. The possibility of having exact analytic results depends heavily on whether the body under consideration possesses some given symmetry (See Chapter XII in Ref.[7]) and how the charge is distributed[8–10]. However, finding the exact analytical equilibrium charge distribution (the one that makes the body an equipotential) is not a simple problem even for a regular body[11]. The few exceptions where the equilibrium charge distribution is known are cases such as that of an infinitely thin two-dimensional (2D) circular conducting disk or the trivial example of a three-dimensional (3D) conducting spherical surface. Curiously enough, there is not an entirely definite answer to the exact nature of the equilibrium charge distribution even in such a simple-looking system as a one-dimensional (1D) straight wire[12–15].

Since the equilibrium charge distribution for the majority of regular bodies (solid cylinder, hollow cylindrical shell, solid cube, square plate, etc.) is impossible to obtain analytically one typically assumes a uniform charge distribution in the hope of facilitating the calculations. The assumption of uniform charge distribution over the volume or surface of a given body may lead to analytic results for particular regular bodies with high spherical or cylindrical symmetry[16]. The mathematical methods to obtain such results are not unique but typical approaches that work well are variants of the 2D or 3D Fourier transform methods for systems with spherical/cylindrical symmetry or suitable mathematical transformations that rely on auxiliary functions for square/rectangular plate systems. For instance, the electrostatic self-energy of a uniformly charged solid cylinder[17], solid cube[18] or square plate[19] was obtained analytically using one or another of the methods mentioned above.

A uniformly charged hemispherical surface stands out within the realm of regular bodies[20–25] since it represents a simple system that lacks the spherical symmetry of its full spherical surface counterpart. For this reason, calculation of its electric field or potential at an arbitrary point in space is very difficult and, we believe, an exact expression is not possible in a compact analytical form. Nevertheless, despite the lack of spherical symmetry, a uniformly charged hemispherical surface still retains the axial symmetry around the azimuthal (longitudinal) angle. This allows us to deduce that the electric field or potential are only functions of radial distance, r and polar angle, θ, but not azimuthal angle, φ and can be generally written, respectively, as $\vec{E}(r,\theta)$ and $V(r,\theta)$ (obviously, with the assumption that a spherical system of coordinates is chosen with origin at the center of the hemispherical surface and $x-y$ plane on the "equator"). One can obtain the electric field and potential at special points like the center of hemispherical surface[26] but we are not aware of compact general analytical results for either of them at an arbitrary location in space. One may be tempted to conclude from this fact that the electrostatic energy stored in a hemispherical surface with uniform surface charge density cannot be calculated exactly given that the electric field or potential are not known analytically.

This work shows that this conclusion is not exactly true. Despite the difficult nature of the problem, we were able to calculate this quantity. The mathematical

approach used to solve the problem is not trivial. The objective of the calculation may look out of reach at various steps including the moment when a complicated infinite sum appears at the end of the process. Nevertheless, the greatest reward comes right at this juncture. It turns out that this infinite sum can be calculated exactly and the final result for the electrostatic potential energy stored in a hemispherical surface with uniform surface charge density is remarkably simple.

II. MODEL AND RESULTS

Let us consider a hemispherical surface with uniform surface charge density. It is assumed that the hemispherical surface has a radius, R and contains a total positive charge, Q that is spread uniformly over its surface. As a result, the uniform surface charge density may be written as:

$$\sigma = \frac{Q}{2\,\pi\,R^2}\ . \tag{1}$$

For simplicity, let us choose a "northern" hemispherical surface. We adopt a spherical system of coordinates with origin at the center of the hemispherical surface and $x-y$ plane on the "equator". For this choice, the surface domain that contains the uniformly distributed charge is represented by:

$$S\ :\ \left\{r=R\ ;\ 0\le\theta\le\frac{\pi}{2}\ ;\ 0\le\varphi<2\,\pi\right\}\ , \tag{2}$$

where θ is the polar angle and φ is the azimuthal (longitudinal) angle. Consider the elementary charges, dQ_i located at position vectors, $\vec{r}_i$ on the hemispherical surface (for such a case, in spherical coordinates, $r_i = R$, $0 \le \theta_i \le \pi/2$, $0 \le \varphi_i < 2\,\pi$). These elementary charges are uniformly spread on the respective elementary surfaces, $dS_i = R^2\,\sin\theta_i\,d\theta_i\,d\varphi_i$ where $i = 1$ and 2. The electrostatic potential energy stored in the body, namely, its electrostatic self-energy is written as:

$$U = \frac{k_e\,\sigma^2}{2}\iint\limits_S dS_1\iint\limits_S dS_2\ \frac{1}{|\vec{r}_1-\vec{r}_2|}\ , \tag{3}$$

where k_e is Coulomb's electric constant, S is the integration domain in Eq.(2) and the factor "2" is needed to avoid double-counting. The quantity in Eq.(3) can be written more explicitly as:

$$U = \frac{k_e\,\sigma^2\,R^4}{2}\int\limits_0^{\pi/2} d\theta_1\ \sin\theta_1\int\limits_0^{2\,\pi} d\varphi_1 \int\limits_0^{\pi/2} d\theta_2\ \sin\theta_2\int\limits_0^{2\,\pi} d\varphi_2\ \frac{1}{|\vec{r}_1-\vec{r}_2|}\ . \tag{4}$$

The calculation of the integral above is not easy. We used various approaches and strategies to calculate it exactly since it is obvious that direct integration does not work. After several attempts, we found out that the only approach that succeded is the one that we report in this work. The scheme that we implement utilizes the axial symmetry of the hemispherical surface in a spherical system of coordinates and employs suitable mathematical transformations that eventually reduce the integral problem into a final infinite series problem. The resulting infinite series is rapidly convergent and can be summed exactly. The only drawback of the process is that several mathematical transformations rely heavily on properties of various special functions such as Legendre polynomials.

We start the calculation of the integral in Eq.(4) by rewriting $1/|\vec{r}_1-\vec{r}_2|$ using the Legendre formula for $r_1 = r_2 = R$ which reads:

$$\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{R}\sum_{l=0}^{\infty} P_l(\cos\gamma)\ , \tag{5}$$

where γ is the angle included between vectors $\vec{r}_1$ and $\vec{r}_2$ and $P_l(\cos\gamma)$ are Legendre polynomials. Appendix C in pg. 598 of Ref.[7] represents a good quick introduction to the main properties of the Legendre polynomials. Note that: $\cos\gamma = \cos\theta_1\,\cos\theta_2 + \sin\theta_1\,\sin\theta_2\,\cos(\varphi_1-\varphi_2)$.

The following addition theorem found in pg. 599 of Ref.[7] holds for Legendre polynomials:

$$P_l(\cos\gamma) = P_l(\cos\,\theta_1)\,P_l(\cos\,\theta_2) + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}\,P_l^m(\cos\,\theta_1)\,P_l^m(\cos\,\theta_2)\,\cos\left[m\,(\varphi_1-\varphi_2)\right]\ , \tag{6}$$

where $P_l^m(\cos\theta)$ are the associated Legendre polynomials. One substitutes the quantity $P_l(\cos\gamma)$ from Eq.(6) into Eq.(5) and integrates the resulting expression over the azimuthal angles φ_1 and φ_2. The terms with $m = 1, 2, \ldots$ do not contribute when one carries out such an integration:

$$\int\limits_0^{2\,\pi} d\varphi_1\int\limits_0^{2\,\pi} d\varphi_2\ \frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{(2\,\pi)^2}{R}\sum_{l=0}^{\infty} P_l(\cos\,\theta_1)\,P_l(\cos\,\theta_2)\ . \tag{7}$$

After substituting the result from Eq.(7) into the expression in Eq.(4) one obtains:

$$U = \frac{k_e\,\sigma^2\,R^4}{2}\,\frac{(2\,\pi)^2}{R}\sum_{l=0}^{\infty}\left\{\left[\int\limits_0^{\pi/2} d\theta_1\ \sin\theta_1\,P_l(\cos\theta_1)\right]\left[\int\limits_0^{\pi/2} d\theta_2\ \sin\theta_2\,P_l(\cos\theta_2)\right]\right\}\ . \tag{8}$$

At this juncture, one introduces the dummy variable $x = \cos\theta_i$ $(i = 1, 2)$ which allows one to write:

$$U = \frac{1}{2}\,\frac{k_e\,Q^2}{R}\sum_{l=0}^{\infty}\left[\int\limits_0^1 dx\,P_l(x)\right]^2\ . \tag{9}$$

The result above can be easily verified after recalling the definition of $\sigma = Q/(2\,\pi\,R^2)$ from Eq.(1) which immediately leads to the expression $\sigma^2\,R^4\,(2\,\pi)^2 = Q^2$. A final analytical result now hinges on our ability to overcome the following last two hurdles: (i) Calculate the integral of a Legendre polynomial over half range, $\int_0^1 dx\,P_l(x) = c(l)$ where $c(l)$ is some non-obvious l-dependent parameter; and (ii) Calculate a potentially infinite sum of l-dependent terms, $\sum_{l=0}^{\infty} c(l)^2$ hoping that such a sum may lead to an analytic result. Having reached this stage, let us simplify the notation and write the quantity of interest as:

$$U = \frac{C}{2}\,\frac{k_e\,Q^2}{R}\ , \tag{10}$$

where

$$C = \sum_{l=0}^{\infty}\left[\int_0^1 dx\,P_l(x)\right]^2\ . \tag{11}$$

Integrals of Legendre polynomials over half range are not routinely found in standard textbooks[27–29]. Therefore, we had to do some extra work to obtain a general formula for the integral, $\int_0^1 dx\,P_n(x)\ ;\ n = 0, 1, 2, \ldots$ where the recurrent formula for the Legendre polynomials was used. Non-zero integrals occur only for Legendre polynomials of order 0 and order $1, 3, 5, \ldots$. The final expressions may be written in a convenient form as:

$$\int_0^1 dx\,P_0(x) = 1\ , \tag{12}$$

and

$$\int_0^1 dx\,P_{2\,l+1}(x) = \frac{(-1)^l}{2^{2\,l+1}\,(l+1)}\,\frac{(2\,l)!}{(l!)^2}\quad ;\quad l = 0, 1, \ldots\ . \tag{13}$$

Note the index $(2\,l+1)$ representing the order of the Legendre polynomials. This notation guarantees that only Legendre polynomials of the form, $P_1(x), P_3(x), \ldots$ are picked up when the indices $l = 0, 1, \ldots$ are selected in Eq.(13). This means that C can be written as:

$$C = 1 + \sum_{l=0}^{\infty}\left[\int_0^1 dx\,P_{2\,l+1}(x)\right]^2\ . \tag{14}$$

One uses the result from Eq.(13) to explicitly express C as an infinite sum:

$$C = 1 + \sum_{l=0}^{\infty}\left[\frac{(-1)^l}{2^{2\,l+1}\,(l+1)}\,\frac{(2\,l)!}{(l!)^2}\right]^2\ . \tag{15}$$

The constant, C is of the order of 1, it is a rapidly convergent sum and can be found numerically. The summation with l up to 50 gives $C = 1.27318$, for l up to 100 gives $C = 1.27322$, and with l up to 200 it results in $C = 1.27324$. The last estimate is very close to $4/\pi$ with precision of $\pm 5 \times 10^{-6}$.

In fact, it became immediately clear that:

$$C = \frac{4}{\pi}\ , \tag{16}$$

when we were pleasantly surprised to find out the following simple result for the following infinite sum:

$$\sum_{l=0}^{\infty}\left[\frac{(-1)^l}{2^{2\,l+1}\,(l+1)}\,\frac{(2\,l)!}{(l!)^2}\right]^2 = \frac{4-\pi}{\pi} = \frac{4}{\pi} - 1\ . \tag{17}$$

The formula above was obtained by using symbolic computation software[30] and the result was checked numerically to a very high degree of accuracy. This allows us to write the quantity in Eq.(10) as:

$$U = \frac{2}{\pi}\,\frac{k_e\,Q^2}{R}\ , \tag{18}$$

where Q represents the total charge spread uniformly on a hemispherical surface with radius R.

Note that the expression in Eq.(8) for a hemispherical surface can be generalized to an arbitrary spherical cap surface of radius R with polar angle θ between 0 and $\theta_{max}(\leq \pi)$. For such an occurrence, we write:

$$U(\theta_{max}) = k_e\,\sigma^2\,R^3\,2\,\pi^2\,C(\theta_{max})\ , \tag{19}$$

where

$$C(\theta_{max}) = \sum_{l=0}^{\infty}\left[\int_0^{\theta_{max}} d\theta\,\sin\theta\,P_l(\cos\theta)\right]^2\ . \tag{20}$$

In this case, the surface charge density appearing in Eq.(19) is not the one in Eq.(1), but is the one given from the following expression:

$$\sigma = \frac{Q}{2\,\pi\,R^2(1-\cos\theta_{max})}\ , \tag{21}$$

where the quantity in the denominator represents the surface area of the spherical cap surface under consideration. The analysis of the dependence of electrostatic self-energy on the angle θ_{max} would be very useful. The integrals in Eq.(20) can be calculated numerically. As special cases, $\theta_{max} = \pi/2$ represents a hemispherical surface while $\theta_{max} = \pi$ corresponds to a full spherical surface. Note that, in this notation, $C(\theta_{max} = \pi/2) = C = 4/\pi \approx 1.27324$ for the case of a hemispherical surface. To illustrate this approach we considered two different angles, $\theta_{max} = \pi/4$ and $\pi/2$ and calculated numerically $C(\theta_{max})$ for increasing values of l. The results for $C(\theta_{max})$ are shown in Table. I. We also verified that the sums converge very well not only for these two values of θ_{max}, but also for few other values limited to the $\pi/10 \leq \theta_{max} \leq \pi$ range.

TABLE I: Numerically calculated values of $C(\theta_{max})$ for increasing values of l ranging from $l = 10$ to $l = 100$. The data have a numerical accuracy of five digits after the decimal point meaning that the fifth digit after the decimal point is rounded.

l	$C(\theta_{max} = \pi/4)$	l	$C(\theta_{max} = \pi/2)$
10	0.19227	10	1.27800
20	0.19296	20	1.27286
30	0.19309	30	1.27307
40	0.19314	40	1.27314
50	0.19317	50	1.27318
100	0.19320	100	1.27322

Furthermore, one can modify Eq.(19) to derive a general formula for the electrostatic energy stored in any given spherical cap surface with arbitrary angle, θ_{max} with the understanding that the total charge, Q is kept constant and uniformly spread over the spherical cap surface:

$$U(\theta_{max}) = \frac{1}{2}\,\frac{C(\theta_{max})}{(1-\cos\theta_{max})^2}\,\frac{k_e\,Q^2}{R}\ . \qquad (22)$$

In Fig. 1 we show the dependence of $U(\theta_{max})$ as a function of θ_{max} for 8 different spherical cap surfaces corresponding to values of angle, $\theta_{max} = (\pi/8)\,n$; $n = 1, \ldots, 8$. The constant $C(\theta_{max})$ is numerically calculated from Eq.(20) by taking $l = 100$. Note that $U(\theta_{max} = \pi/2)$ corresponds to the result in Eq.(18) for a uniformly charged hemispherical surface. One also notices that $U(\theta_{max} = \pi) = 0.5\,k_e\,Q^2/R$ which is the known result for the electrostatic self-energy of a uniformly charged full spherical surface.

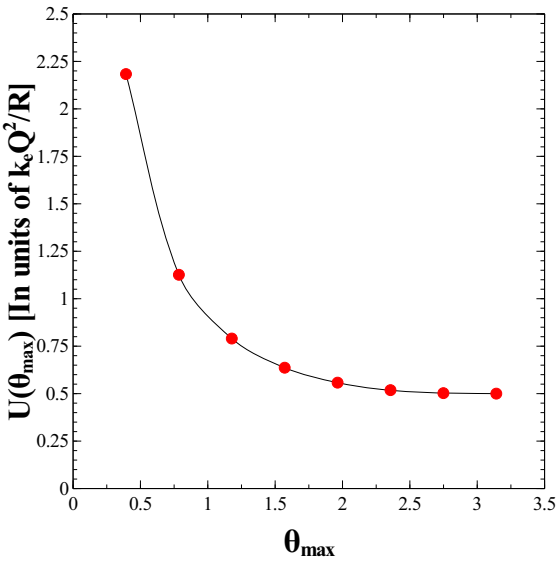

FIG. 1: Values of the stored electrostatic self-energy, $U(\theta_{max})$ as a function of θ_{max} for 8 different spherical cap surfaces with angle θ_{max} ranging from $\pi/8$ to π (Filled solid circles). The spherical cap surfaces are uniformly charged with total charge, Q kept constant. The energy is expressed in units of k_eQ^2/R. Note that $U(\theta_{max} = \pi) = 0.5\,k_eQ^2/R$ is the exact known value of the electrostatic potential energy stored in a uniformly charged spherical surface. The constant $C(\theta_{max})$ was calculated numerically from the corresponding sum in Eq.(20) by taking $l = 100$ in all cases considered. The solid line joining the data points is a guide for the eyes.

III. DISCUSSION AND CONCLUSIONS

In this work we obtained an exact result for the electrostatic energy stored in a hemispherical surface with uniform surface charge density. A uniformly charged hemispherical surface with all the charge uniformly spread on its surface represents a very challenging problem to solve since its lacks the spherical symmetry of a uniformly charged spherical surface or a solid sphere. Despite these challenges, it was established that an analytic solution is possible if one uses a combination of suitable mathematical transformations appropriate for the axial symmetry of the problem within the framework of a spherical system of coordinates.

We started the calculation of the energy by employing a very useful mathematical transformation that applies to a standard Coulomb interaction potential in a spherical system of coordinates and involves Legendre polynomials. One may not see any considerable reduction of the difficulty of the problem by doing so, but at least, this transformation removes the cumbersome Coulomb term from the denominator. By proceeding further, one sees that the method used allows one to obtain a general expression of the electrostatic energy as an infinite sum of terms involving integrals of Legendre polynomials over their half range. Integrals of Legendre polynomials over their half range are not routine. As a result, we had to obtain an exact analytic result for such integrals. This allowed us to write the expression of the electrostatic energy as an infinite sum. Luckily, this infinite sum could be calculated exactly with the final result that is quite simple and compact.

One can employ the same technique to calculate the electrostatic potential and field around a hemispherical surface or a spherical cap surface. Based on the symmetry of the problem, both potential and field can be expressed, respectively, as functions: $V(r,\theta)$ and $\vec{E}(r,\theta) = -\vec{\nabla}\,V(r,\theta)$. Analytical results may be possible for special cases, for instance, along the z axis ($\theta = 0$), but in general the final expressions should be numerically calculated. Knowing the electric field around a charged surface of a given shape, for instance around a spherical cap surface, turns out to be an important question to answer. For example, space-based radio antennas in spacecrafts become negatively charged and this occurrence affects the accuracy of radio-wave measurements.

Therefore, adjustments for the electric field coming from charges on the antennas should be made. Such antennas are made of conductors and have a shape that is generally a paraboloid of revolution. Nevertheless, one may crudely model them as a uniformly charged spherical cap surface and, thus, calculate numerically the electric field in the surrounding.

A discussion on how the electrostatic self-energy of a conducting hemispherical surface compares to that with uniform charge distribution for the same radius and same charge is also helpful since it shows to the reader the intricacies of the theory of the potential. Calculating the equilibrium charge distribution on a conducting hemispherical surface, namely, obtaining the precise analytic form of the surface charge density that leads to an equipotential surface (same potential all over the hemispherical surface) is an unsolvable analytical problem. There are very few regular conducting bodies (apart a spherical surface or solid sphere) for which this problem is analytically solvable. A conducting disk is a rare example where an exact solution is available. It is known that the equilibrium surface charge density of a charged conducting disk is strikingly different from that of a uniformly charged disk. Nevertheless, the electrostatic self-energy of a conducting disk does not differ much from its uniformly charged counterpart with a relative energy difference between them of about 8 % in percentage[22]. Based on general physical considerations and based on our knowledge of the case study of a conducting disk, we expect the electrostatic self-energy stored in a conducting hemispherical surface with equlibrium charge distribution to be smaller than, but likely not very different from, the value found for the corresponding uniformly charged hemispherical surface.

The result obtained for the electrostatic energy stored in a hemispherical surface with uniform surface charge density is important in its own merit and can be useful to other scientific disciplines. For instance, such a result can help computational physicists to gauge the accuracy of various theoretical approximations and numerical methods used in computational physical sciences since the calculation of the electrostatic self-energy of a hemispherical surface with uniform surface charge density is not a simple task for numerical computational methods[31,32]. In this sense, we believe that the results reported here may be of interest to the specialized, as well as to the broad audiences of researchers working in the field.

Acknowledgments

This research was supported in part by National Science Foundation (NSF) grant no. DMR-2001980.

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